LIMITING FRACTIONAL AND LORENTZ SPACES ESTIMATES OF DIFFERENTIAL FORMS

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ABSTRACT. We obtain estimates in Besov, Lizorkin-Triebel and Lorentz spaces of differential forms on $\mathbf{R}^n$ in terms of their L^1 norm.

1. INTRODUCTION

The classical Hodge theory states that if $u \in C_c^\infty(\mathbf{R}^N; \bigwedge^\ell \mathbf{R}^n)$, if $1 < p < \infty$, one has

$$\|Du\|_{L^p} \le C(\|du\|_{\mathrm{L}^p} + \|\delta u\|_{\mathrm{L}^p}) \tag{1}$$

where du is the exterior differential and δu the exterior codifferential. This estimate is known to fail when $p = 1$ or $p = \infty$.

When $p = 1$, J. Bourgain and H. Brezis [2, 3], and L. Lanzani and E. Stein [5] have obtained for $2 \le \ell \le n - 2$ the estimate

$$\|u\|_{L^{n/(n-1)}} \le C(\|du\|_{\mathrm{L}^1} + \|\delta u\|_{\mathrm{L}^1}),$$

which would be the consequence that would follow by the Sobolev embedding from (1) with $p = 1$. When $\ell = 1$ or $\ell = n - 1$ one has to assume that du or δu vanishes.

I. Mitrea and M. Mitrea [6] have in a recent work extended these estimates to homogeneous Besov spaces. Using interpolation theory, they could replace the norm $\|u\|_{L^{n/(n-1)}}$ by $\|u\|_{\dot{B}^s_{p,q}}$ with $\frac{1}{p} - \frac{s}{n} = 1 - \frac{1}{n}$ and $q = \frac{2}{1-s}$. The goal of the present paper is to improve the assumption on q by relying on previous results and methods.

We follow H. Triebel [10] for the definitions of the function spaces. The first result is the estimate for the Besov spaces $\dot{B}^s_{p,q}(\mathbf{R}^n)$:

Theorem 1. *For every $s \in (0, 1)$, $p > 1$ and $q > 1$, if*

$$\frac{1}{p} - \frac{s}{n} = 1 - \frac{1}{n}, \tag{2}$$

then there exists $C > 0$ such that for every $u \in C_c^\infty(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$, with moreover, $\delta u = 0$ if $\ell = 1$ and $du = 0$ if $\ell = n - 1$, one has

$$\|u\|_{\dot{B}^s_{p,q}} \le C(\|du\|_{L^1} + \|\delta u\|_{L^1}).$$

In particular, since $\|u\|_{\dot{W}^{s,p}} = \|u\|_{\dot{B}^s_{p,p}}$, one has the estimate

$$\|u\|_{\dot{W}^{s,p}} \le C(\|du\|_{L^1} + \|\delta u\|_{L^1}). \tag{3}$$

In Theorem 1, we assume that $q > 1$. If it held for some $q \in (0, 1]$, then the embedding of $F^1_{1,2}(\mathbf{R}^n) \subset \dot{B}^0_{n/(n-1),q}(\mathbf{R}^n)$ would hold. This can only be the case

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when $q \geq 1$. Therefore, the only possible improvement of Theorem 1 would be the limiting case $q = 1$:

Open problem 1. Does Theorem 1 hold for $q = 1$?

The estimate of Theorem 1 follows from the corresponding estimate for homogeneous Lizorkin–Triebel spaces $\dot{F}^s_{p,q}(\mathbf{R}^n)$:

Theorem 2. *For every $s \in (0,1)$, $p > 1$ and $q > 0$, if* (2) *holds, then there exists $C > 0$ such that for every $u \in C^\infty_c(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$, with moreover, $\delta u = 0$ if $\ell = 1$ and $du = 0$ if $\ell = n-1$, one has*

$$\|u\|_{\dot{F}^s_{p,q}} \leq C(\|du\|_{L^1} + \|\delta u\|_{L^1}).$$

Note that here there is no restriction on $q > 0$. Finally, the latter estimate has an interesting consequence for Lorentz spaces.

Theorem 3. *For every $q > 1$, then there exists $C > 0$ such that for every $u \in C^\infty_c(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$, with moreover, $\delta u = 0$ if $\ell = 1$ and $du = 0$ if $\ell = n-1$, one has*

$$\|u\|_{L^{\frac{n}{n-1},q}} \leq C(\|du\|_{L^1} + \|\delta u\|_{L^1}).$$

In Theorem 3 the case $q = 1$ and $\ell = 0$ is equivalent with the embedding of $W^{1,1}(\mathbf{R}^n)$ in $L^{\frac{n}{n-1},1}(\mathbf{R}^n)$ which was obtained by J. Peetre [8] (see also [15]). This raises the question

Open problem 2. Does Theorem 3 hold for $q = 1$ and $\ell \geq 1$?

The proof of the theorems rely on the techniques developed by the author [11,12], and on classical embeddings and regularity theory in fractional spaces.

2. The main tool

Our main tool is a generalization of an estimate for divergence-free L^1 vector fields of the author [11]:

Proposition 2.1. *For every $s \in (0,1)$, $p > 1$ and $q > 0$ with $sp = n$, there exists $C > 0$ such that for every $f \in (C^\infty_c \cap L^1)(\mathbf{R}^n; \bigwedge^{n-1} \mathbf{R}^n)$ and $\varphi \in C^\infty_c(\mathbf{R}^n; \bigwedge \mathbf{R}^n)$, if $df = 0$,*

$$\int_{\mathbf{R}^n} f \wedge \varphi \leq C \|f\|_{L^1} \|\varphi\|_{\dot{F}^s_{p,q}}.$$

The proof of this proposition follows the method introduced by the author [4, 11, 12, 14] and followed subsequently by L. Lanzani and E. Stein [5] and I. Mitrea and M. Mitrea [6]. The extension to the case $q = p$ in a previous work of the author [11, Remark 5] (see also [14, Remark 2] and [4]); the proposition can be deduced therefrom by following a remark in a subsequent paper [13, Remark 4.2].

Proof. Write $\varphi = \varphi^1 dx_1 + \varphi^n dx_n$ and $f = f_1 dx_2 \wedge \dots \wedge dx_n + \dots + f_n dx_1 \wedge \dots \wedge dx_{n-1}$. Without loss of generality, we shall estimate

$$\int_{\mathbf{R}^n} f_1 \varphi^1.$$

Fix $t \in \mathbf{R}$, and consider the function $\psi : \mathbf{R}^{n-1} \to \mathbf{R}$ defined by $\psi(y) = \varphi^1(t, y)$. Choose $\rho \in C^\infty_c(\mathbf{R}^n)$ such that $\int_{\mathbf{R}^{n-1}} \rho = 1$ and set $\rho_\varepsilon(y) = \frac{1}{\varepsilon^{n-1}} \rho(\frac{x}{\varepsilon})$. For every $\alpha \in (0,1)$, there is a constant $C > 0$ that only depends on ρ and α such that

$$\|\nabla \rho_\varepsilon * \psi\|_{L^\infty} \leq C \varepsilon^{\alpha-1} |\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}$$

and

$$\|\psi - \rho_\varepsilon * \psi\|_{L^\infty} \leq C \varepsilon^{\alpha} |\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})},$$

where $|\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}$ is the $C^{0,\alpha}$ seminorm of ψ, i.e.,

$$|\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})} = \sup_{y,z\in\mathbf{R}^{n-1}} \frac{|\psi(z)-\psi(y)|}{|z-y|}.$$

One has on the one hand

$$\int_{\mathbf{R}^{n-1}} f_1(t,\cdot)(\psi-\rho_\varepsilon * v) \le C\|f_1(t,\cdot)\|_{L^1(\mathbf{R}^{n-1})}\varepsilon^{\alpha}|\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}.$$

On the other hand, by integration by parts, and since $\sum_{i=1}^n \partial_i f_i = 0$,

$$\begin{aligned}\int_{\mathbf{R}^{n-1}} f_1(t,\cdot)\rho_\varepsilon * \psi &= -\sum_{i=2}^n \int_{\mathbf{R}^{n-1}}\int_{\mathbf{R}^+} f_i(t,y)\partial_i(\rho_\varepsilon*\psi)(y)\,dt\,dy\\ &\le C\|f_1(t,\cdot)\|_{L^1(\mathbf{R}^{n-1})}\varepsilon^{\alpha-1}|\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}.\end{aligned}$$

Taking $\varepsilon = \|f\|_{L^1(\mathbf{R}^n)}/\|f(t,\cdot)\|_{L^1(\mathbf{R}^{n-1})}$, one obtains

$$\int_{\mathbf{R}^{n-1}} f_1\psi \le C\|f\|^{\alpha}_{L^1(\mathbf{R}^n)}\|f_1(t,\cdot)\|^{1-\alpha}_{L^1(\mathbf{R}^{n-1})}|\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}. \tag{4}$$

Now, by the embedding theorem for Lizorkin–Triebel spaces, one has the estimate

$$|\psi|_{C^{0,\alpha}} \le C\|\psi\|_{\dot F^s_{p,q}(\mathbf{R}^{n-1})};$$

with $\alpha = \frac{1}{p}$; hence from (4) we deduce the inequality

$$\int_{\mathbf{R}^{n-1}} f_1\psi \le C\|f\|^{\frac{1}{p}}_{L^1}\|f_1(\cdot,t)\|^{1-\frac{1}{p}}_{L^1}\|\psi\|_{\dot F^s_{p,q}(\mathbf{R}^{n-1})}.$$

Now, recalling that, as a direct consequence of the Fubini property that is stated in [10, Theorem 2.5.13] [1, Théorème 2], [9, Theorem 2.3.4/2]

$$\Big(\int_{\mathbf{R}} \|\varphi(t,\cdot)\|^p_{\dot F^{s,p}(\mathbf{R}^{n-1})}\,dt\Big) \le C\|\varphi\|^p_{\dot F^{s,p}(\mathbf{R}^n)},$$

one concludes, using Hölder's inequality that

$$\int_{\mathbf{R}^n} f_1 u^1 \le C\|f\|^{\frac{1}{p}}_{L^1}\int_{\mathbf{R}}\big(\|f_1(\cdot,t)\|^{1-\frac{1}{p}}_{L^1}\|\varphi(t,\cdot)\|_{\dot F^{s,p}(\mathbf{R}^{n-1})}\big)\,dt \le C'\|f\|_{L^1}\|\varphi\|_{\dot F^{s,p}(\mathbf{R}^n)}.$$

□

Proposition 2.2. *For every $s\in(0,1)$, $p>1$ with $\frac{1}{p}+\frac{s}{n}=1$, $q>1$ and $1\le \ell\le n-1$, there exists $C>0$ such that for every $f\in C^\infty_c(\mathbf{R}^n;\bigwedge^\ell\mathbf{R}^n)$ with $df=0$, one has*

$$\|f\|_{\dot F^{-s,p}_q} \le C\|f\|_{L^1}.$$

Proof. The proposition will be proved by downward induction. The proposition is true for $\ell = n-1$ by Proposition 2.1. Assume now that it holds for $\ell+1$, and let $f\in C^\infty_c(\mathbf{R}^n;\bigwedge^\ell\mathbf{R}^n)$. Since $d(f\wedge dx_i)=0$, Proposition 2.1 is applicable and

$$\|f\|_{\dot F^{-s,\frac{n}{n-s}}_q} \le \sum_{i=1}^n \|f\wedge dx_i\|_{\dot F^{-s,\frac{n}{n-s}}_q} \le C\sum_{i=1}^n\|f\|_{L^1} = Cn\|f\|_{L^1}.$$ □

A useful corollary of the previous proposition is

Corollary 2.3. *For every $s\in(0,1)$, $p>1$ with $\frac{1}{p}+\frac{s}{n}=1$, $q>1$ and $1\le\ell\le n-1$, there exists $C>0$ such that for every $f\in C^\infty_c(\mathbf{R}^n;\bigwedge^\ell\mathbf{R}^n)$ with $df=0$, one has*

$$\|f\|_{\dot B^{-s,p}_q}\le C\|f\|_{L^1}.$$

Proof. This follows from classical embeddings between Besov and Lizorkin–Triebel spaces; see the proof of Theorem 1 below. □

3. Proofs of the main results

We begin by proving Theorem 2:

Proof of Theorem 2. To fix ideas, assume that $2 \le \ell \le n-1$. Recall that one has

$$u = d\big(K * (\delta u)\big) + \delta\big(K * (du)\big),$$

where the Newton kernel is defined by $K(x) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}|x|^{n-2}}$. By the classical elliptic estimates for Lizorkin–Triebel spaces,

$$\|K * (\delta u)\|_{\dot{F}^{s+1}_{p,q}} \le C\|\delta u\|_{\dot{F}^{s-1}_{p,q}} \qquad \text{and} \qquad \|K * (\delta u)\|_{\dot{F}^{s+1}_{p,q}} \le C\|\delta u\|_{\dot{F}^{s-1}_{p,q}}.$$

Now, since $d(du) = 0$, Proposition 2.2 is applicable and yields

$$\|K * (du)\|_{\dot{F}^{s+1}_{p,q}} \le C\|\delta u\|_{L^1}.$$

Since $\delta(\delta u) = 0$, one can by the Hodge duality between d and δ treat $\|K*(du)\|_{\dot{F}^{s+1}_{p,q}}$ similarly. □

We can now deduce Theorem 1 from Theorem 2:

Proof of Theorem 1. First assume that $q \ge p$. Then one has

$$\|u\|_{\dot{B}^s_{p,q}} \le C\|u\|_{\dot{F}^s_{p,q}},$$

and Theorem 1 follows from Theorem 2. Otherwise, if $q < p$, then by the embedding theorems of Besov spaces,

$$\|u\|_{\dot{B}^s_{p,q}} \le C\|u\|_{\dot{B}^r_{q,q}} = C\|u\|_{\dot{F}^r_{q,q}}$$

with $r = s + n(\frac{1}{q} - \frac{1}{p})$ and Theorem 1 also follows from Theorem 2. □

We finish with the proof of Theorem 3. It relies on the

Lemma 3.1. *For every $s > 0$, $p > 1$ and $q > 1$ with $sq < n$ and*

$$\frac{1}{p} = \frac{1}{q} - \frac{s}{n}, \tag{5}$$

there exists $C > 0$ such that for every $u \in C^\infty_c(\mathbf{R}^n)$,

$$\|u\|_{L^{p,q}} \le C\|u\|_{\dot{F}^s_{q,2}}.$$

Proof. One has

$$u = I_s * ((-\Delta)^{\frac{s}{2}} u),$$

where the Riesz kernel I_s is defined for $x \in \mathbf{R}^n$ by

$$I_s(x) = \frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}} 2^s \Gamma(\frac{s}{2})|x|^{n-s}}.$$

One has then by Sobolev inequality for Riesz potentials in Lorentz spaces of R. O'Neil [7] (see also e.g. [15, Theorem 2.10.2]),

$$\|u\|_{L^{r,p}} \le C\|(-\Delta)^{\frac{s}{2}} u\|_{L^p}.$$

One concludes by noting that $\|(-\Delta)^{\frac{s}{2}} u\|_{L^p}$ and $\|u\|_{\dot{F}^s_{p,2}}$ are equivalent norms [10, Theorem 2.3.8 and section 5.2.3]. □

Proof of Theorem 3. Choose s so that (5) holds with $p = \frac{n}{n-1}$. Since $\frac{1}{q} - \frac{s}{n} = 1 - \frac{1}{n}$, one can combine Theorem 2 and Lemma 3.1 to obtain the conclusion. □

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